

Generalized Convex Functions and Their Applications

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Abstract : This study focuses on convex functions and their generalized. Thus, we start this study by giving the definition of convex functions and some of their properties and discussing a simple geometric property. Then we generalize E-convex functions and establish some their properties. Moreover, we give generalized s -convex functions in the second sense and present some new inequalities of generalized Hermite-Hadamard type for the class of functions whose second local fractional derivatives of order α in absolute value at certain powers are generalized s -convex functions in the second sense. At the end, some examples that these inequalities are able to be applied to some special means are showed.

1 Introduction

Let $M \subseteq \mathbb{R}$ be an interval. A function $\varphi : M \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a convex if for any $y_1, y_2 \in M$ and $\eta \in [0, 1]$,

$$\varphi(\eta y_1 + (1 - \eta)y_2) \leq \eta\varphi(y_1) + (1 - \eta)\varphi(y_2). \quad (1.1)$$

If the inequality (1.1) is the strict inequality, then φ is called a strict convex function.

From a geometrical point of view, a function φ is convex provided that the line segment connecting any two points of its graph lies on or above the graph. The function φ is strictly convex provided that the line segment

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connecting any two points of its graph lies above the graph. If $-\varphi$ is convex (resp. strictly convex), then φ is called concave (resp. strictly concave).

The convexity of functions have been widely used in many branches of mathematics, for example in mathematical analysis, function theory, functional analysis, optimization theory and so on. For a production function $x = \varphi(L)$, concavity of φ is expressed economically by saying that φ exhibits diminishing returns. While if φ is convex, then it exhibits increasing returns. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions, see([2, 12, 23]).

Recently, the fractal theory has received significantly remarkable attention from scientists and engineers. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension[10, 19]. Many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches [2, 6, 31]. Particularly, in [27], Yang stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus and the monotonicity of function.

Throughout this chapter \mathbb{R}^α will be denoted a real linear fractal set.

Definition 1.1. [15] A function $\varphi: M \subset \mathbb{R} \rightarrow \mathbb{R}^\alpha$ is called generalized convex if

$$\varphi(\eta y_1 + (1 - \eta)y_2) \leq \eta^\alpha \varphi(y_1) + (1 - \eta)^\alpha \varphi(y_2) \quad (1.2)$$

for all $y_1, y_2 \in M$, $\eta \in [0, 1]$ and $\alpha \in (0, 1]$.

It is called strictly generalized convex if the inequality (1.2) holds strictly whenever y_1 and y_2 are distinct points and $\eta \in (0, 1)$. If $-\varphi$ is generalized convex (respectively, strictly generalized convex), then φ is generalized concave (respectively, strictly generalized concave).

In $\alpha = 1$, we have a convex function ,i.e, (1.1) is obtained.

Let $f \in {}_{a_1}I_{a_2}^{(\alpha)}$ be a generalized convex function on $[a_1, a_2]$ with $a_1 < a_2$. Then,

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{\Gamma(1 + \alpha)}{(a_2 - a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leq \frac{f(a_1) + f(a_2)}{2^\alpha}. \quad (1.3)$$

is known as generalized Hermite-Hadamard's inequality [14]. Many authors paid attention to the study of generalized Hermite-Hadamard's inequality and generalized convex function, see [4, 16]. If $\alpha = 1$ in (1.3), then [8]

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}, \quad (1.4)$$

which is known as classical Hermite-Hadamard inequality, for more properties about this inequality we refer the interested readers to [9, 13].

2 Generalized E-convex Functions

In 1999, Youness [30] introduced E-convexity of sets and functions, which have some important applications in various branches of mathematical sciences [1, 20]. However, Yang [26] showed that some results given by Youness [30] seem to be incorrect. Chen [7] extended E-convexity to a semi E-convexity and discussed some of its properties. For more results on E-convex function or semi E-convex function see [3, 11, 17, 18, 25].

Definition 2.1. [30]

- (i) A set $B \subseteq \mathbb{R}^n$ is called a *E-convex* iff there exists $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\eta E(r_1) + (1 - \eta)E(r_2) \in B, \forall r_1, r_2 \in B, \eta \in [0, 1].$$

- (ii) A function $g: \mathbb{R}^n \longrightarrow \mathbb{R}$ is called *E-convex (ECF)* on a set $B \subseteq \mathbb{R}^n$ iff there exists $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and

$$g(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \eta g(E(r_1)) + (1 - \eta)g(E(r_2)), \forall r_1, r_2 \in B, \eta \in [0, 1].$$

The following propositions were proved in [30]:

Proposition 2.2. (i) Suppose that a set $B \subseteq \mathbb{R}^n$ is *E-convex*, then $E(B) \subseteq B$.

- (ii) Assume that $E(B)$ is convex and $E(B) \subseteq B$, then B is *E-convex*.

Definition 2.3. A function $g: \mathbb{R}^n \longrightarrow \mathbb{R}^\alpha$ is called a *generalized E-convex function (gECF)* on a set $B \subseteq \mathbb{R}^n$ iff there exists a map $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that B is an *E-convex* set and

$$g(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)), \quad (2.1)$$

$\forall r_1, r_2 \in B, \eta \in (0, 1)$ and $\alpha \in (0, 1]$ On the other hand, if

$$g(\eta E(x_1) + (1 - \eta)E(x_2)) \geq \eta^\alpha g(E(x_1)) + (1 - \eta)^\alpha g(E(x_1)),$$

$\forall x_1, x_2 \in B, \eta \in (0, 1)$ and $\alpha \in (0, 1]$, then g is called *generalized E-concave* on B . If the inequality sings in the previous two inequality are strict, then g is called *generalized strictly E-convex* and *generalized strictly E-concave*, respectively.

Proposition 2.4. (i) Every ECF on a convex set B is gECF, where $E = I$.

(ii) If $\alpha = 1$ in equation (2.1), then g is called ECF on a set B .

(iii) If $\alpha = 1$ and $E = I$ in equation (2.1), then g is called a convex function

The following two examples show that generalized E-convex function which are not necessarily generalized convex.

Example 2.5. Assume that $B \subseteq \mathbb{R}^2$ is given as

$$B = \{(x_1, x_2) \in \mathbb{R}^2: \mu_1(0, 0) + \mu_2(0, 3) + \mu_3(2, 1)\},$$

with $\mu_i > 0, \sum_{i=1}^3 \mu_i = 1$ and define a map $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such as $E(x_1, x_2) = (0, x_2)$. The function $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^\alpha$ defined by

$$g(x_1, x_2) = \begin{cases} x_1^{3\alpha}; x_2 < 1, \\ x_1^\alpha x_2^{3\alpha}; x_2 \geq 1 \end{cases}$$

The function g is gECF on B , but is not generalized convex.

Remark 2.6. If $\alpha \longrightarrow 0$ in the above example, then g goes to generalized convex function.

Example 2.7. Assume that $g: \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ is defined as

$$g(r) = \begin{cases} 1^\alpha; r > 0, \\ (-r)^\alpha; r \leq 0 \end{cases}$$

and assume that $E: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $E(r) = -r^2$. Hence, \mathbb{R} is an E-convex set and g is gECF, but is not generalized convex.

Theorem 2.1. Assume that $B \subseteq \mathbb{R}^n$ is an E -convex set and $g_1: B \rightarrow \mathbb{R}$ is an ECF. If $g_2: U \rightarrow \mathbb{R}^\alpha$ is non-decreasing generalized convex function such that the rang $g_1 \subset U$, then $g_2 \circ g_1$ is a gECF on B .

Proof. Since g_1 is ECF, then

$$g_1(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \eta g_1(E(r_1)) + (1 - \eta)g_1(E(r_2)),$$

$\forall r_1, r_2 \in B$ and $\eta \in [0, 1]$. Also, since g_2 is non-decreasing generalized convex function, then

$$\begin{aligned} g_2 \circ g_1(\eta E(r_1) + (1 - \eta)E(r_2)) &\leq g_2[\eta g_1(E(r_1)) + (1 - \eta)g_1(E(r_2))] \\ &\leq \eta^\alpha g_2(g_1(E(r_1))) + (1 - \eta)^\alpha g_2(g_1(E(r_2))) \\ &= \eta^\alpha g_2 \circ g_1(E(r_1)) + (1 - \eta)^\alpha g_2 \circ g_1(E(r_2)) \end{aligned}$$

which implies that $g_2 \circ g_1$ is a gECF on B .

Similarly, $g_2 \circ g_1$ is a strictly gECF if g_2 is a strictly non-decreasing generalized convex function. \square

Theorem 2.2. Assume that $B \subseteq \mathbb{R}^n$ is an E -convex set, and $g_i: B \rightarrow \mathbb{R}^\alpha, i = 1, 2, \dots, l$ are generalized E -convex function. Then,

$$g = \sum_{i=1}^l k_i^\alpha g_i$$

is a generalized E -convex on B for all $k_i^\alpha \in \mathbb{R}^\alpha$

Proof. Since $g_i, i = 1, 2, \dots, l$ are gECF, then

$$g_i(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \eta^\alpha g_i(E(r_1)) + (1 - \eta)^\alpha g_i(E(r_2)),$$

$\forall r_1, r_2 \in B, \eta \in [0, 1]$ and $\alpha \in (0, 1]$. Then,

$$\begin{aligned} &\sum_{i=1}^l k_i^\alpha g_i(\eta E(r_1) + (1 - \eta)E(r_2)) \\ &\leq \eta^\alpha \sum_{i=1}^l k_i^\alpha g_i(E(r_1)) + (1 - \eta)^\alpha \sum_{i=1}^l k_i^\alpha g_i(E(r_2)) \\ &= \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)). \end{aligned}$$

Thus, g is a gECF. \square

Definition 2.8. Assume that $B \subseteq \mathbb{R}^n$ is a convex set. A function $g: B \longrightarrow \mathbb{R}^\alpha$ is called *generalized quasi convex* if

$$g(\eta r_1 + (1 - \eta)r_2) \leq \max \{g(r_1), g(r_2)\},$$

$\forall r_1, r_2 \in B$ and $\eta \in [0, 1]$.

Definition 2.9. Assume that $B \subseteq \mathbb{R}^n$ is an E -convex set. A function $g: B \longrightarrow \mathbb{R}^\alpha$ is called

(i) *Generalized E -quasiconvex function* iff

$$g(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \max \{g(E(r_1)), E(g(r_2))\},$$

$\forall r_1, r_2 \in B$ and $\eta \in [0, 1]$.

(ii) *Strictly generalized E -quasiconcave function* iff

$$g(\eta E(r_1) + (1 - \eta)E(r_2)) > \min \{g(E(r_1)), E(g(r_2))\},$$

$\forall r_1, r_2 \in B$ and $\eta \in [0, 1]$.

Theorem 2.3. Assume that $B \subseteq \mathbb{R}^n$ is an E -convex set, and $g_i: B \longrightarrow \mathbb{R}^\alpha, i = 1, 2, \dots, l$ are $gECF$. Then,

(i) The function $g: B \longrightarrow \mathbb{R}^\alpha$ which is defined by $g(r) = \sup_{i \in I} g_i(r), r \in B$ is a $gECF$ on B .

(ii) If $g_i, i = 1, 2, \dots, l$ are generalized E -quasiconvex functions on B , then the function g is a generalized E -quasiconvex function on B .

Proof. (i) Due to $g_i, i \in I$ be $gECF$ on B , then

$$\begin{aligned} & g(\eta E(r_1) + (1 - \eta)E(r_2)) \\ &= \sup_{i \in I} g_i(\eta E(r_1) + (1 - \eta)E(r_2)) \\ &\leq \eta^\alpha \sup_{i \in I} g_i(E(r_1)) + (1 - \eta)^\alpha \sup_{i \in I} g_i(E(r_2)) \\ &= \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)). \end{aligned}$$

Hence, g is a $gECF$ on B .

(ii) Since $g_i, i \in I$ are generalized E-quasiconvex functions on B , then

$$\begin{aligned}
g(\eta E(x_1) + (1 - \eta)E(x_2)) &= \sup_{i \in I} g_i(\eta E(x_1) + (1 - \eta)E(x_2)) \\
&\leq \sup_{i \in I} \max \{g_i(E(x_1)), g_i(E(x_2))\} \\
&= \max \left\{ \sup_{i \in I} g_i(E(x_1)), \sup_{i \in I} g_i(E(x_2)) \right\} \\
&= \max \{g(E(x_1)), g(E(x_2))\}.
\end{aligned}$$

Hence, g is a generalized E-quasiconvex function on B . □

Considering $B \subseteq \mathbb{R}^n$ is a nonempty E-convex set. From Propostion 2.2(i), we get $E(B) \subseteq B$. Hence, for any $g: B \rightarrow \mathbb{R}^\alpha$, the restriction $\tilde{g}: E(B) \rightarrow \mathbb{R}^\alpha$ of g to $E(B)$ defined by

$$\tilde{g}(\tilde{x}) = g(\tilde{x}), \forall \tilde{x} \in E(B)$$

is well defined.

Theorem 2.4. Assume that $B \subseteq \mathbb{R}^n$, and $g: B \rightarrow \mathbb{R}^\alpha$ is a generalized E-quasiconvex function on B . Then, the restriction $\tilde{g}: U \rightarrow \mathbb{R}^\alpha$ of g to any nonempty convex subset U of $E(B)$ is a generalized quasiconvex on U .

Proof. Assume that $x_1, x_2 \in U \subseteq E(B)$, then there exist $x_1^*, x_2^* \in B$ such that $x_1 = E(x_1^*)$ and $x_2 = E(x_2^*)$. Since U is a convex set, we have

$$\eta x_1 + (1 - \eta)x_2 = \eta E(x_1^*) + (1 - \eta)E(x_2^*) \in U, \forall \eta \in [0, 1].$$

Therefore, we have

$$\begin{aligned}
\tilde{g}(\eta x_1 + (1 - \eta)x_2) &= \tilde{g}(\eta E(x_1^*) + (1 - \eta)E(x_2^*)) \\
&\leq \max \{g(E(x_1^*)), g(E(x_2^*))\} \\
&= \max \{g(x_1), g(x_2)\} \\
&= \max \{\tilde{g}(x_1), \tilde{g}(x_2)\}.
\end{aligned}$$
□

Theorem 2.5. Assume that $B \subseteq \mathbb{R}^n$ is an E-convex set, and $E(B)$ is a convex set. Then, $g: B \rightarrow \mathbb{R}^\alpha$ is a generalized E-quasiconvex on B iff its restriction $\tilde{g} = g|_{E(B)}$ is a generalized quasiconvex function on $E(B)$.

Proof. Due to Theorem2.4, the if condition is true. Conversely, suppose that $x_1, x_2 \in B$, then $E(x_1), E(x_2) \in E(B)$ and $\eta E(x_1) + (1 - \eta)E(x_2) \in E(B) \subseteq B, \forall \eta \in [0, 1]$. Since $E(B) \subseteq B$, then

$$\begin{aligned} g(\eta E(x_1) + (1 - \eta)E(x_2)) &= \tilde{g}(\eta E(x_1) + (1 - \eta)E(x_2)) \\ &\leq \max \{ \tilde{g}(E(x_1)), \tilde{g}(E(x_2)) \} \\ &= \max \{ g(E(x_1)), g(E(x_2)) \}. \end{aligned}$$

□

An analogous result to Theorem2.4 for the generalized E-convex case is as follows:

Theorem 2.6. Assume that $B \subseteq \mathbb{R}^n$ is an E-convex set, and $g: B \rightarrow \mathbb{R}^\alpha$ is a gECF on B . Then, the restriction $\tilde{g}: U \rightarrow \mathbb{R}^\alpha$ of g to any nonempty convex subset U of $E(B)$ is a gCF.

An analogous result to Theorem2.5 for the generalized E-convex case is as follows:

Theorem 2.7. Assume that $B \subseteq \mathbb{R}^n$ is an E-convex set, and $E(B)$ is a convex set. Then, $g: B \rightarrow \mathbb{R}^\alpha$ is a gECF on B iff its restriction $\tilde{g} = g|_{E(B)}$ is a gCF on $E(B)$.

The lower level set of $goE: B \rightarrow \mathbb{R}^\alpha$ is defined as

$$L_{r^\alpha}(goE) = \{x \in B: (goE)(x) = g(E(x)) \leq r^\alpha, r^\alpha \in \mathbb{R}^\alpha\}.$$

The lower level set of $\tilde{g}: E(B) \rightarrow \mathbb{R}^\alpha$ is defined as

$$L_{r^\alpha}(\tilde{g}) = \{\tilde{x} \in E(B): \tilde{g}(\tilde{x}) = g(\tilde{x}) \leq r^\alpha, r^\alpha \in \mathbb{R}^\alpha\}.$$

Theorem 2.8. Suppose that $E(B)$ be a convex set. A function $g: B \rightarrow \mathbb{R}^\alpha$ is a generalized E-quasiconvex iff $L_{r^\alpha}(\tilde{g})$ of its restriction $\tilde{g}: E(B) \rightarrow \mathbb{R}^\alpha$ is a convex set for each $r^\alpha \in \mathbb{R}^\alpha$.

Proof. Due to $E(B)$ be a convex set, then for each $E(x_1), E(x_2) \in E(B)$, we have $\eta E(x_1) + (1 - \eta)E(x_2) \in E(B) \subseteq B$. Let $\tilde{x}_1 = E(x_1)$ and $\tilde{x}_2 = E(x_2)$.

If $\tilde{x}_1, \tilde{x}_2 \in L_{r^\alpha}(\tilde{g})$, then $g(\tilde{x}_1) \leq r^\alpha$ and $g(\tilde{x}_2) \leq r^\alpha$. Thus,

$$\begin{aligned}
\tilde{g}(\eta\tilde{x}_1 + (1-\eta)\tilde{x}_2) &= g(\eta\tilde{x}_1 + (1-\eta)\tilde{x}_2) \\
&= g(\eta E(x_1) + (1-\eta)E(x_2)) \\
&\leq \max\{g(E(x_1)), g(E(x_2))\} \\
&= \max\{g(\tilde{x}_1), g(\tilde{x}_2)\} \\
&= \max\{\tilde{g}(\tilde{x}_1), \tilde{g}(\tilde{x}_2)\} \\
&\leq r^\alpha.
\end{aligned}$$

which show that $\eta\tilde{x}_1 + (1-\eta)\tilde{x}_2 \in L_{r^\alpha}(\tilde{g})$. Hence, $L_{r^\alpha}(\tilde{g})$ is a convex set.

Conversely, let $L_{r^\alpha}(\tilde{g})$ be a convex set for each $r^\alpha \in \mathbb{R}^\alpha$, i.e., $\eta\tilde{x}_1 + (1-\eta)\tilde{x}_2 \in L_{r^\alpha}(\tilde{g}), \forall \tilde{x}_1, \tilde{x}_2 \in L_{r^\alpha}(\tilde{g})$ and $r^\alpha = \max\{g(\tilde{x}_1), g(\tilde{x}_2)\}$. Thus,

$$\begin{aligned}
g(\eta E(x_1) + (1-\eta)E(x_2)) &= \tilde{g}(\eta E(x_1) + (1-\eta)E(x_2)) \\
&= \tilde{g}(\eta\tilde{x}_1 + (1-\eta)\tilde{x}_2) \\
&\leq r^\alpha \\
&= \max\{g(\tilde{x}_1), g(\tilde{x}_2)\} \\
&= \max\{g(E(x_1)), g(E(x_2))\}.
\end{aligned}$$

Hence, g is a generalized E-quasiconvex. \square

Theorem 2.9. Let $B \subseteq \mathbb{R}^n$ be a nonempty E-convex set and let $g_1: B \rightarrow \mathbb{R}^\alpha$ be a generalized E-quasiconvex on B . Suppose that $g_2: \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$ is a non-decreasing function. Then, $g_2 \circ g_1$ is a generalized E-quasiconvex.

Proof. Since $g_1: B \rightarrow \mathbb{R}^\alpha$ is generalized E-quasiconvex on B and $g_2: \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$ is a non-decreasing function, then

$$\begin{aligned}
(g_2 \circ g_1)(\eta E(x_1) + (1-\eta)E(x_2)) &= g_2(g_1(\eta E(x_1) + (1-\eta)E(x_2))) \\
&\leq g_2(\max\{g_1(E(x_1)), g_1(E(x_2))\}) \\
&= \max\{(g_2 \circ g_1)(E(x_1)), (g_2 \circ g_1)(E(x_2))\}
\end{aligned}$$

which shows that $g_2 \circ g_1$ is a generalized E-quasiconvex on B . \square

Theorem 2.10. *If the function g is a gECF on $B \subseteq \mathbb{R}^n$, then g is a generalized E -quasiconvex on B .*

Proof. Assume that g is a gECF on B . Then,

$$\begin{aligned} g(\eta E(r_1) + (1 - \eta)E(r_2)) &\leq \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)) \\ &\leq \eta^\alpha \max \{g(E(r_1)), g(E(r_2))\} \\ &\quad + (1 - \eta)^\alpha \max \{g(E(r_1)), g(E(r_2))\} \\ &= \max \{g(E(r_1)), g(E(r_2))\}. \end{aligned}$$

□

3 E^α -epigraph

Definition 3.1. *Assume that $B \subseteq \mathbb{R}^n \times \mathbb{R}^\alpha$ and $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$, then the set B is called E^α -convex set iff*

$$(\eta E(x_1) + (1 - \eta)E(x_2), \eta^\alpha r_1^\alpha + (1 - \eta)^\alpha r_2^\alpha) \in B$$

$$\forall (x_1, r_1^\alpha), (x_2, r_2^\alpha) \in B, \eta \in [0, 1] \text{ and } \alpha \in (0, 1].$$

Now, the E^α -epigraph of g is given by

$$epi_{E^\alpha}(g) = \{(E(x), r^\alpha) : x \in B, r^\alpha \in \mathbb{R}^\alpha, g(E(x)) \leq r^\alpha\}.$$

A sufficient condition for g to be a gECF is given by the following theorem:

Theorem 3.1. *Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an idempotent map. Assume that $B \subseteq \mathbb{R}^n$ is an E -convex set and $epi_{E^\alpha}(g)$ is an E^α -convex set where $g: B \rightarrow \mathbb{R}^\alpha$, then g is a gECF on B .*

Proof. Assume that $r_1, r_2 \in B$ and $(E(r_1), g(E(r_1))), (E(r_2), g(E(r_2))) \in epi_{E^\alpha}(g)$. Since $epi_{E^\alpha}(g)$ is E^α -convex set, we have

$$(\eta E(E(r_1)) + (1 - \eta)E(E(r_2)), \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2))) \in epi_{E^\alpha}(g),$$

then

$$g(\eta E(E(r_1)) + (1 - \eta)E(E(r_2))) \leq \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)).$$

Due to $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an idempotent map, then

$$g(\eta E(r_1) + (1 - \eta)E(r_2)) \leq \eta^\alpha g(E(r_1)) + (1 - \eta)^\alpha g(E(r_2)).$$

Hence, g is a gECF. □

Theorem 3.2. Assume that $\{B_i\}_{i \in I}$ is a family of E^α -convex sets. Then, their intersection $\cap_{i \in I} B_i$ is an E^α -convex set.

Proof. Considering $(x_1, r_1^\alpha), (x_2, r_2^\alpha) \in \cap_{i \in I} B_i$, then $(x_1, r_1^\alpha), (x_2, r_2^\alpha) \in B_i$, $\forall i \in I$. By E^α -convexity of $B_i, \forall i \in I$, then we have

$$(\eta E(x_1) + (1 - \eta)E(x_2), \eta^\alpha r_1^\alpha + (1 - \eta)^\alpha r_2^\alpha) \in B_i,$$

$\forall \eta \in [0, 1]$ and $\alpha \in (0, 1]$. Hence,

$$(\eta E(x_1) + (1 - \eta)E(x_2), \eta^\alpha r_1^\alpha + (1 - \eta)^\alpha r_2^\alpha) \in \cap_{i \in I} B_i.$$

□

The following theorem is a special case of Theorem 2.3(i) where $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an idempotent map.

Theorem 3.3. Assume that $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an idempotent map, and $B \subseteq \mathbb{R}^n$ is an E -convex set. Let $\{g_i\}_{i \in I}$ be a family function which have bounded from above. If $\text{epi}_{E^\alpha}(g_i)$ are E^α -convex sets, then the function g which defined by $g(x) = \sup_{i \in I} g_i(x), x \in B$ is a $gECF$ on B .

Proof. Since

$$\text{epi}_{E^\alpha}(g_i) = \{(E(x), r^\alpha): x \in B, r^\alpha \in \mathbb{R}^\alpha, g_i(E(x)) \leq r^\alpha, i \in I\}$$

are E^α -convex set in $B \times \mathbb{R}^\alpha$, then

$$\begin{aligned} \cap_{i \in I} \text{epi}_{E^\alpha}(g_i) &= \{(E(x), r^\alpha): x \in B, r^\alpha \in \mathbb{R}^\alpha, g_i(E(x)) \leq r^\alpha, i \in I\} \\ &= \{(E(x), r^\alpha): x \in B, r^\alpha \in \mathbb{R}^\alpha, g(E(x)) \leq r^\alpha\}, \end{aligned} \quad (3.1)$$

where $g(E(x)) = \sup_{i \in I} g_i(E(x))$, also is E^α -convex set. Hence, $\cap_{i \in I} \text{epi}_{E^\alpha}(g_i)$ is an E^α -epigraph, then by Theorem 3.2, g is a generalized E -convex function on B . □

4 Generalized s -convex functions

There are many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches see [5, 24, 28]

In [14], two kinds of generalized s -convex functions on fractal sets are introduced as follows:

Definition 4.1. (i) A function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$, is called a generalized s -convex ($0 < s < 1$) in the first sense if

$$\varphi(\eta_1 y_1 + \eta_2 y_2) \leq \eta_1^{s\alpha} \varphi(y_1) + \eta_2^{s\alpha} \varphi(y_2) \quad (4.1)$$

for all $y_1, y_2 \in \mathbb{R}_+$ and all $\eta_1, \eta_2 \geq 0$ with $\eta_1^s + \eta_2^s = 1$, this class of functions is denoted by GK_s^1 .

(ii) A function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$, is called a generalized s -convex ($0 < s < 1$) in the second sense if (4.1) holds for all $y_1, y_2 \in \mathbb{R}_+$ and all $\eta_1, \eta_2 \geq 0$ with $\eta_1 + \eta_2 = 1$, this class of functions is denoted by GK_s^2 .

In the same paper, [14], Mo and Sui proved that all functions which are generalized s -convex in the second sense, for $s \in (0, 1)$, are non-negative.

If $\alpha = 1$ in Definition 4.1, then we have the classical s -convex functions in the first sense (second sense) see [8].

Also, in [8], Dragomir and Fitzpatrick demonstrated a variation of Hadamard's inequality which holds for s -convex functions in the second sense.

Theorem 4.2. Assume that $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a s -convex function in the second sense, $0 < s < 1$ and $y_1, y_2 \in \mathbb{R}_+$, $y_1 < y_2$. If $\varphi \in L^1([y_1, y_2])$, then

$$2^{s-1} \varphi\left(\frac{y_1 + y_2}{2}\right) \leq \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \varphi(z) dz \leq \frac{\varphi(y_1) + \varphi(y_2)}{s + 1}. \quad (4.2)$$

If we set $k = \frac{1}{s+1}$, then it is the best possible in the second inequality in (4.2).

A variation of generalized Hadamard's inequality which holds for generalized s -convex functions in the second sense [2].

Theorem 4.3. Assume that $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+^\alpha$ is a generalized s -convex function in the second sense where $0 < s < 1$ and $y_1, y_2 \in \mathbb{R}_+$ with $y_1 < y_2$. If $\varphi \in L^1([y_1, y_2])$, then

$$\begin{aligned} 2^{\alpha(s-1)} \varphi\left(\frac{y_1 + y_2}{2}\right) &\leq \frac{\Gamma(1 + \alpha)}{(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \\ &\leq \frac{\Gamma(1 + s\alpha)\Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)} (\varphi(y_1) + \varphi(y_2)). \end{aligned} \quad (4.3)$$

Proof. We know that φ is generalized s -convex in the second sense, which lead to

$$\varphi(\eta y_1 + (1 - \eta)y_2) \leq \eta^{\alpha s} \varphi(y_1) + (1 - \eta)^{\alpha s} \varphi(y_2), \forall \eta \in [0, 1].$$

Then the following inequality can be written:

$$\begin{aligned} \Gamma(1 + \alpha) {}_0I_1^{(\alpha)} \varphi(\eta y_1 + (1 - \eta)y_2) &\leq \varphi(y_1) \Gamma(1 + \alpha) {}_0I_1^{(\alpha)} \eta^{\alpha s} \\ &\quad + \varphi(y_2) \Gamma(1 + \alpha) {}_0I_1^{(\alpha)} (1 - \eta)^{\alpha s} \\ &= \frac{\Gamma(1 + s\alpha) \Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)} (\varphi(y_2) + y_2) \end{aligned}$$

By considering $z = \eta y_1 + (1 - \eta)y_2$. Then

$$\begin{aligned} \Gamma(1 + \alpha) {}_0I_1^{(\alpha)} \varphi(\eta y_1 + (1 - \eta)y_2) &= \frac{\Gamma(1 + \alpha)}{(y_1 - y_2)^\alpha} {}_{y_2}I_{y_1}^{(\alpha)} \varphi(x) \\ &= \frac{\Gamma(1 + \alpha)}{(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(z). \end{aligned}$$

Here

$$\frac{\Gamma(1 + \alpha)}{(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \leq \frac{\Gamma(1 + s\alpha) \Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)} (\varphi(y_1) + \varphi(y_2)).$$

Then, the second inequality in (4.3) is given.

Now

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \frac{\varphi(z_1) + \varphi(z_2)}{2^{\alpha s}}, \forall z_1, z_2 \in I. \quad (4.4)$$

Let $z_1 = \eta y_1 + (1 - \eta)y_2$ and $z_2 = (1 - \eta)y_1 + \eta y_2$ with $\eta \in [0, 1]$.

Hence, by applying (4.4), the next inequality holds

$$\varphi\left(\frac{y_1 + y_2}{2}\right) \leq \frac{\varphi(\eta y_1 + (1 - \eta)y_2) + \varphi((1 - \eta)y_1 + \eta y_2)}{2^{\alpha s}}, \forall \eta \in [0, 1].$$

So

$$\frac{1}{\Gamma(1 + \alpha)} \int_0^1 \varphi\left(\frac{y_1 + y_2}{2}\right) (d\eta)^\alpha \leq \frac{1}{2^{\alpha(s-1)}(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(z)$$

Then,

$$2^{\alpha(s-1)} \varphi\left(\frac{y_1 + y_2}{2}\right) \leq \frac{\Gamma(1 + \alpha)}{(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(z).$$

□

Lemma 4.4. Assume that $\varphi: [y_1, y_2] \subset \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ is a local fractional derivative of order α ($\varphi \in D_\alpha$) on (y_1, y_2) with $y_1 < y_2$. If $\varphi^{(2\alpha)} \in C_\alpha[y_1, y_2]$, then the following equality holds:

$$\begin{aligned} & \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) - \frac{\Gamma(1+2\alpha)}{2^\alpha}\varphi\left(\frac{y_1+y_2}{2}\right) \\ &= \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[{}_0I_1^{(\alpha)}\gamma^{2\alpha}\varphi^{(2\alpha)}\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right. \\ & \quad \left. + {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}\varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma)\frac{y_1+y_2}{2}\right) \right] \end{aligned}$$

Proof. From the local fractional integration by parts, we get

$$\begin{aligned} B_1 &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \gamma^{2\alpha} \varphi^{(2\alpha)}\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) (d\gamma)^\alpha \\ &= \left(\frac{2}{y_2-y_1}\right)^\alpha \varphi^{(\alpha)}\left(\frac{y_1+y_2}{2}\right) \\ & \quad - \Gamma(1+2\alpha) \left(\frac{2}{y_2-y_1}\right)^{2\alpha} \gamma^\alpha \varphi\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \Big|_0^1 \\ & \quad + \Gamma(1+2\alpha)\Gamma(1+\alpha) \left(\frac{2}{b-a}\right)^{2\alpha} \int_0^1 \varphi\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) (d\gamma)^\alpha \\ &= \left(\frac{2}{y_2-y_1}\right)^\alpha \varphi^{(\alpha)}\left(\frac{y_1+y_2}{2}\right) - \Gamma(1+2\alpha) \left(\frac{2}{y_2-y_1}\right)^{2\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) \\ & \quad + \Gamma(1+2\alpha)\Gamma(1+\alpha) \left(\frac{2}{y_2-y_1}\right)^{2\alpha} \int_0^1 \varphi\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) (d\gamma)^\alpha \end{aligned}$$

Setting $x = \gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1$, for $\gamma \in [0, 1]$ and multiply the both sides in the last equation by $\frac{(y_2-y_1)^{2\alpha}}{16^\alpha}$, we get

$$\begin{aligned} B_1 &= \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} {}_0I_1^{(\alpha)}\gamma^{2\alpha}\varphi^{(2\alpha)}\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \\ &= \frac{(y_2-y_1)^\alpha}{8^\alpha} \varphi^{(\alpha)}\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)}{4^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) \\ & \quad + \frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{2^\alpha(y_2-y_1)^\alpha} \int_{y_1}^{\frac{y_1+y_2}{2}} \varphi(x)(dx)^\alpha. \end{aligned}$$

By the similar way, also we have

$$\begin{aligned}
B_2 &= \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} {}_0 I_1^{(\alpha)} (\gamma - 1)^{2\alpha} \varphi^{(2\alpha)} \left(\gamma y_2 + (1 - \gamma) \frac{y_1 + y_2}{2} \right) \\
&= -\frac{(y_2 - y_1)^\alpha}{8^\alpha} \varphi^{(\alpha)} \left(\frac{y_1 + y_2}{2} \right) - \frac{\Gamma(1 + 2\alpha)}{4^\alpha} \varphi \left(\frac{y_1 + y_2}{2} \right) \\
&\quad + \frac{\Gamma(1 + 2\alpha)\Gamma(1 + \alpha)}{2^\alpha (y_2 - y_1)^\alpha} \int_{\frac{y_1 + y_2}{2}}^{y_2} \varphi(x) (dx)^\alpha.
\end{aligned}$$

Thus, adding B_1 and B_2 , we get the desired result. \square

Theorem 4.5. Assume that $\varphi : U \subset [0, \infty) \rightarrow \mathbb{R}^\alpha$ such that $\varphi \in D_\alpha$ on $\text{Int}(U)$ ($\text{Int}(U)$ is the interior of U) and $\varphi^{(2\alpha)} \in C_\alpha[y_1, y_2]$, where $y_1, y_2 \in U$ with $y_1 < y_2$. If $|\varphi|$ is generalized s -convex on $[y_1, y_2]$, for some fixed $0 < s \leq 1$, then the following inequality holds:

$$\begin{aligned}
&\left| \frac{\Gamma(1 + 2\alpha)}{2^\alpha} \varphi \left(\frac{y_1 + y_2}{2} \right) - \frac{\Gamma(1 + 2\alpha)[\Gamma(1 + \alpha)]^2}{2^\alpha (y_2 - y_1)^\alpha} {}_{y_1} I_{y_2}^{(\alpha)} \varphi(x) \right| \\
&\leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^\alpha \Gamma(1 + (s + 2)\alpha)}{\Gamma(1 + (s + 3)\alpha)} \left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right| + \left[\frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \right. \right. \\
&\quad \left. \left. - 2^\alpha \frac{\Gamma(1 + (s + 1)\alpha)}{\Gamma(1 + (s + 2)\alpha)} + \frac{\Gamma(1 + (s + 2)\alpha)}{\Gamma(1 + (s + 3)\alpha)} \right] \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\} \quad (4.5)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^{\alpha(2-s)} \Gamma(1 + (s + 2)\alpha)}{\Gamma(1 + (s + 3)\alpha)} \frac{\Gamma(1 + s\alpha)\Gamma(1 + \alpha)}{\Gamma(1 + (s + 1)\alpha)} + \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \right. \\
&\quad \left. - \frac{2^\alpha \Gamma(1 + (s + 1)\alpha)}{\Gamma(1 + (s + 2)\alpha)} + \frac{\Gamma(1 + (s + 2)\alpha)}{\Gamma(1 + (s + 3)\alpha)} \right\} \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right]. \quad (4.6)
\end{aligned}$$

Proof. From Lemma 4.4, we have

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[{}_0I_1^{(\alpha)} \gamma^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma \frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right| \right. \\
& \quad \left. + {}_0I_1^{(\alpha)} (\gamma-1)^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma) \frac{y_1+y_2}{2}\right) \right| \right] \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} {}_0I_1^{(\alpha)} \gamma^{2\alpha} \left[\gamma^{\alpha s} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| + (1-\gamma)^{\alpha s} \left| \varphi^{(2\alpha)}(y_1) \right| \right] \\
& \quad + \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} {}_0I_1^{(\alpha)} (\gamma-1)^{2\alpha} \left[\gamma^{\alpha s} \left| \varphi^{(2\alpha)}(y_2) \right| + (1-\gamma)^{\alpha s} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| \right] \\
& = \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| \right. \\
& \quad + \left[\frac{\Gamma(1+\alpha s)}{\Gamma(1+(s+1)\alpha)} - 2^\alpha \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)} + \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \right] \left| \varphi^{(2\alpha)}(y_1) \right| \Big\} \\
& \quad + \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| \right. \\
& \quad + \left[\frac{\Gamma(1+\alpha s)}{\Gamma(1+(s+1)\alpha)} - 2^\alpha \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)} + \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \right] \left| \varphi^{(2\alpha)}(y_2) \right| \Big\} \\
& \quad + \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| \Big\} \\
& = \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^\alpha \Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| + \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \right. \right. \\
& \quad \left. \left. - 2^\alpha \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)} + \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+3)\alpha)} \right] \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\}.
\end{aligned}$$

This proves inequality (4.5). Since

$$2^{\alpha(s-1)} \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left(\varphi^{(2\alpha)}(y_1) + \varphi^{(2\alpha)}(y_2) \right),$$

then

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^\alpha\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \frac{2^{-\alpha(s-1)}\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right. \\
& \quad \left. + \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} - \frac{2^\alpha\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)} + \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \right] \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\} \\
& = \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^{\alpha(2-s)}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \right. \\
& \quad \left. - \frac{2^\alpha\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)} + \frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)} \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\}
\end{aligned}$$

Thus, we get the inequality (4.6) and the proof is complete. \square

Remark 4.6. 1. When $\alpha = 1$, Theorem 4.5 reduce to Theorem 2 in [21].

2. If $s = 1$ in Theorem 4.5, then

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^\alpha\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right| + \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right. \right. \\
& \quad \left. \left. - 2^\alpha \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \right] \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\} \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ \frac{2^\alpha\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \frac{[\Gamma(1+\alpha)]^2}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right. \\
& \quad \left. - \frac{2^\alpha\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \right\} \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \quad (4.7)
\end{aligned}$$

3. If $s = 1$ and $\alpha = 1$ in Theorem 4.5, then

$$\begin{aligned}
& \left| \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{1}{y_2-y_1} \int_{y_1}^{y_2} \varphi(x) dx \right| \\
& \leq \frac{(y_2-y_1)^2}{192} \left\{ 6 \left| \varphi''\left(\frac{y_1+y_2}{2}\right) \right| + \left| \varphi''(y_1) \right| + \left| \varphi''(y_2) \right| \right\} \\
& \leq \frac{(y_2-y_1)^2}{48} \{ \left| \varphi''(y_1) \right| + \left| \varphi''(y_2) \right| \}
\end{aligned}$$

We give a new upper bound of the left generalized Hadamard's inequality for generalized s -convex functions in the following theorem:

Theorem 4.7. Assume that $\varphi : U \subset [0, \infty) \rightarrow \mathbb{R}^\alpha$ such that $\varphi \in D_\alpha$ on $\text{Int}(U)$ and $\varphi^{(2\alpha)} \in C_\alpha[y_1, y_2]$, where $y_1, y_2 \in U$ with $y_1 < y_2$. If $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -convex on $[y_1, y_2]$, for some fixed $0 < s \leq 1$ and $p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\ & \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \right]^{\frac{1}{p_2}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \quad \times \left[\left(\left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_1) \right|^{p_2} \right)^{\frac{1}{p_2}} \right. \\ & \quad \left. + \left(\left| \varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_2) \right|^{p_2} \right)^{\frac{1}{p_2}} \right]. \quad (4.8) \end{aligned}$$

Proof. Let $p_1 > 1$, then from Lemma 4.4 and using generalized Hölder's inequality [27], we obtain

$$\begin{aligned} & \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\ & \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left\{ {}_0I_1^{(\alpha)}\gamma^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right| \right. \\ & \quad \left. + {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma)\frac{y_1+y_2}{2}\right) \right| \right\} \\ & \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left({}_0I_1^{(\alpha)}\gamma^{2p_1\alpha} \right)^{\frac{1}{p_1}} \\ & \quad \times \left({}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma\frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right|^{p_2} \right)^{\frac{1}{p_2}} \\ & \quad + \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left({}_0I_1^{(\alpha)}(1-\gamma)^{2p_1\alpha} \right)^{\frac{1}{p_1}} \\ & \quad \times \left({}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma)\frac{y_1+y_2}{2}\right) \right|^{p_2} \right)^{\frac{1}{p_2}}. \end{aligned}$$

Since $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -convex, then

$$\begin{aligned} & {}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)} \left(\gamma \frac{y_1 + y_2}{2} + (1 - \gamma)y_1 \right) \right|^{p_2} \\ & \leq \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2} + \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \left| \varphi^{(2\alpha)}(y_1) \right|^{p_2}, \end{aligned}$$

which means

$$\begin{aligned} & {}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)} \left(\gamma y_2 + (1 - \gamma) \frac{y_1 + y_2}{2} \right) \right|^{p_2} \\ & \leq \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \left| \varphi^{(2\alpha)}(y_2) \right|^{p_2} \\ & \quad + \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2}. \end{aligned}$$

Hence

$$\begin{aligned} & \left| \frac{\Gamma(1 + 2\alpha)}{2^\alpha} \varphi \left(\frac{y_1 + y_2}{2} \right) - \frac{\Gamma(1 + 2\alpha)[\Gamma(1 + \alpha)]^2}{2^\alpha(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\ & \leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s + 1)\alpha)} \right]^{\frac{1}{p_2}} \left[\frac{\Gamma(1 + 2p_1\alpha)}{\Gamma(1 + (2p_1 + 1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \quad \times \left\{ \left[\left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_1) \right|^{p_2} \right]^{\frac{1}{p_2}} \right. \\ & \quad \left. + \left[\left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_2) \right|^{p_2} \right]^{\frac{1}{p_2}} \right\}. \end{aligned}$$

The proof is complete. \square

Remark 4.8. If $s = 1$ in Theorem 4.7, then

$$\begin{aligned} & \left| \frac{\Gamma(1 + 2\alpha)}{2^\alpha} \varphi \left(\frac{y_1 + y_2}{2} \right) - \frac{\Gamma(1 + 2\alpha)[\Gamma(1 + \alpha)]^2}{2^\alpha(y_2 - y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\ & \leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1 + \alpha)}{\Gamma(1 + 2\alpha)} \right]^{\frac{1}{p_2}} \left[\frac{\Gamma(1 + 2p_1\alpha)}{\Gamma(1 + (2p_1 + 1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \quad \times \left\{ \left[\left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_1) \right|^{p_2} \right]^{\frac{1}{p_2}} \right. \\ & \quad \left. + \left[\left| \varphi^{(2\alpha)} \left(\frac{y_1 + y_2}{2} \right) \right|^{p_2} + \left| \varphi^{(2\alpha)}(y_2) \right|^{p_2} \right]^{\frac{1}{p_2}} \right\}. \quad (4.9) \end{aligned}$$

Corollary 4.9. Assume that $\varphi : U \subset [0, \infty) \longrightarrow \mathbb{R}^\alpha$ such that $\varphi \in D_\alpha$ on $\text{Int}(U)$ and $\varphi^{(2\alpha)} \in C_\alpha[y_1, y_2]$, where $y_1, y_2 \in U$ with $y_1 < y_2$. If $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -convex on $[y_1, y_2]$, for some fixed $0 < s \leq 1$ and $p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\ & \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \times \left\{ \left[\left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) + \Gamma(1+(s+1)\alpha) \right)^{\frac{1}{p_2}} \right. \right. \\ & \left. \left. + 2^{\frac{\alpha(1-s)}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}} \right] \left[\left| \varphi^{(2\alpha)}(y_1) \right| + \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\} \end{aligned}$$

Proof. Since $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -convex, then

$$2^{\alpha(s-1)}\varphi^{(2\alpha)}\left(\frac{y_1+y_2}{2}\right) \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left(\varphi^{(2\alpha)}(y_1) + \varphi^{(2\alpha)}(y_2) \right).$$

Hence using (4.8), we get

$$\begin{aligned} & \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)}\varphi(x) \right| \\ & \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \times \left\{ \left[\left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) + \Gamma(1+(s+1)\alpha) \right) |\varphi^{(2\alpha)}(y_1)|^{p_2} \right. \right. \\ & \left. \left. + 2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) |\varphi^{(2\alpha)}(y_2)|^{p_2} \right]^{\frac{1}{p_2}} \right. \\ & \left. + \left[2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) |\varphi^{(2\alpha)}(y_1)|^{p_2} \right. \right. \\ & \left. \left. + \left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) + \Gamma(1+(s+1)\alpha) \right) |\varphi^{(2\alpha)}(y_2)|^{p_2} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

and since $\sum_{i=1}^k (x_i + z_i)^{\alpha n} \leq \sum_{i=1}^k x_i^{\alpha n} + \sum_{i=1}^k z_i^{\alpha n}$

for $0 < n < 1, x_i, z_i \geq 0; \forall 1 \leq i \leq k$, then we have

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\
& \times \left\{ \left[\left(2^{\alpha(1-s)} \Gamma(1+s\alpha) \Gamma(1+\alpha) + \Gamma(1+(s+1)\alpha) \right)^{\frac{1}{p_2}} \left| \varphi^{(2\alpha)}(y_1) \right| \right. \right. \\
& \quad \left. \left. + 2^{\frac{\alpha(1-s)}{p_2}} [\Gamma(1+s\alpha)]^{\frac{1}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}} \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right. \\
& \quad \left. + \left[2^{\frac{\alpha(1-s)}{p_2}} [\Gamma(1+s\alpha)]^{\frac{1}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}} \left| \varphi^{(2\alpha)}(y_1) \right| \right. \right. \\
& \quad \left. \left. + \left(2^{\alpha(1-s)} \Gamma(1+s\alpha) \Gamma(1+\alpha) + \Gamma(1+(s+1)\alpha) \right)^{\frac{1}{p_2}} \left| \varphi^{(2\alpha)}(y_2) \right| \right] \right\}
\end{aligned}$$

where $0 < \frac{1}{p_2} < 1$ for $p_2 > 1$. By a simple calculation, we obtain the required result. \square

Now, the generalized Hadamard's type inequality for generalized s -concave functions.

Theorem 4.10. Assume that $\varphi : U \subset [0, \infty) \rightarrow \mathbb{R}^\alpha$ such that $\varphi \in D_\alpha$ on $\text{Int}(U)$ and $\varphi^{(2\alpha)} \in C_\alpha[y_1, y_2]$, where $y_1, y_2 \in U$ with $y_1 < y_2$. If $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -convex on $[y_1, y_2]$, for some fixed $0 < s \leq 1$ and $p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then the following inequality holds:

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\
& \leq \frac{2^{\frac{\alpha(s-1)}{p_2}} (y_2-y_1)^{2\alpha}}{16^\alpha (\Gamma(1+\alpha))^{\frac{1}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\
& \times \left[\left| \varphi^{(2\alpha)}\left(\frac{3y_1+y_2}{4}\right) \right| + \left| \varphi^{(2\alpha)}\left(\frac{y_1+3y_2}{4}\right) \right| \right]
\end{aligned}$$

Proof. From Lemma 4.4 and using the generalized Hölder inequality for $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$, we get

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha) [\Gamma(1+\alpha)]^2}{2^\alpha (a_2-a_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[{}_0I_1^{(\alpha)} \gamma^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma \frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right| \right. \\
& \quad \left. + {}_0I_1^{(\alpha)} (\gamma-1)^{2\alpha} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma) \frac{y_1+y_2}{2}\right) \right| \right] \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left({}_0I_1^{(\alpha)} \gamma^{2p_1\alpha} \right)^{\frac{1}{p_1}} \left({}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma \frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right|^{p_2} \right)^{\frac{1}{p_2}} \\
& \quad + \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left({}_0I_1^{(\alpha)} (\gamma-1)^{2p_1\alpha} \right)^{\frac{1}{p_1}} \left({}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma) \frac{y_1+y_2}{2}\right) \right|^{p_2} \right)^{\frac{1}{p_2}}.
\end{aligned}$$

Since $|\varphi^{(2\alpha)}|^{p_2}$ is generalized s -concave, then

$${}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma \frac{y_1+y_2}{2} + (1-\gamma)y_1\right) \right|^{p_2} \leq \frac{2^{\alpha(s-1)}}{\Gamma(1+\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{3y_1+y_2}{4}\right) \right|^{p_2} \quad (4.10)$$

also

$${}_0I_1^{(\alpha)} \left| \varphi^{(2\alpha)}\left(\gamma y_2 + (1-\gamma) \frac{y_1+y_2}{2}\right) \right|^{p_2} \leq \frac{2^{\alpha(s-1)}}{\Gamma(1+\alpha)} \left| \varphi^{(2\alpha)}\left(\frac{y_1+3y_2}{4}\right) \right|^{p_2} \quad (4.11)$$

From (4.10) and (4.11), we observe that

$$\begin{aligned}
& \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi\left(\frac{y_1+y_2}{2}\right) - \frac{\Gamma(1+2\alpha) [\Gamma(1+\alpha)]^2}{2^\alpha (y_2-y_1)^\alpha} {}_{y_1}I_{y_2}^{(\alpha)} \varphi(x) \right| \\
& \leq \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \frac{2^{\frac{\alpha(s-1)}{p_2}}}{(\Gamma(1+\alpha))^{\frac{1}{p_2}}} \left| \varphi^{(2\alpha)}\left(\frac{3y_1+y_2}{4}\right) \right| \\
& \quad + \frac{(y_2-y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \frac{2^{\frac{\alpha(s-1)}{p_2}}}{(\Gamma(1+\alpha))^{\frac{1}{p_2}}} \left| \varphi^{(2\alpha)}\left(\frac{y_1+3y_2}{4}\right) \right| \\
& = \frac{2^{\frac{\alpha(s-1)}{p_2}} (y_2-y_1)^{2\alpha}}{16^\alpha (\Gamma(1+\alpha))^{\frac{1}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \left[\left| \varphi^{(2\alpha)}\left(\frac{3y_1+y_2}{4}\right) \right| + \left| \varphi^{(2\alpha)}\left(\frac{y_1+3y_2}{4}\right) \right| \right]
\end{aligned}$$

the proof is complete. \square

Remark 4.11. 1. If $\alpha = 1$ in Theorem 4.10, then

$$\left| \varphi \left(\frac{y_1 + y_2}{2} \right) - \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \varphi(x) dx \right| \\ \leq \frac{2^{\frac{s-1}{q}} (y_2 - y_1)^2}{16} \left[\frac{1}{\Gamma(2p_1 + 1)} \right]^{\frac{1}{p_1}} \left[\left| \varphi'' \left(\frac{3y_1 + y_2}{4} \right) \right| + \left| \varphi'' \left(\frac{y_1 + 3y_2}{4} \right) \right| \right]$$

2. If $s = 1$ and $\frac{1}{3} < \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} < 1, p_1 > 1$ in Theorem 4.10, then

$$\left| \frac{\Gamma(1+2\alpha)}{2^\alpha} \varphi \left(\frac{y_1 + y_2}{2} \right) - \frac{\Gamma(1+2\alpha) [\Gamma(1+\alpha)]^2}{2^\alpha (y_2 - y_1)^\alpha} {}_{y_1} I_{y_2}^{(\alpha)} \varphi(x) \right| \\ \leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha (\Gamma(1+\alpha))^{\frac{1}{p_2}}} \left[\left| \varphi^{(2\alpha)} \left(\frac{3y_1 + y_2}{4} \right) \right| + \left| \varphi^{(2\alpha)} \left(\frac{y_1 + 3y_2}{4} \right) \right| \right]$$

5 Applications to special means

As in [22], some generalized means are considered such as :

$$A(y_1, y_2) = \frac{y_1^\alpha + y_2^\alpha}{2^\alpha}, y_1, y_2 \geq 0;$$

$$L_n(y_1, y_2) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left(y_2^{(n+1)\alpha} - y_1^{(n+1)\alpha} \right) \right]^{\frac{1}{n}}, n \in \mathbb{Z} \setminus \{-1, 0\}, y_1, y_2 \in \mathbb{R}, y_1 \neq y_2.$$

In [14], the following example was given:

let $0 < s < 1$ and $y_1^\alpha, y_2^\alpha, y_3^\alpha \in \mathbb{R}^\alpha$. Defining for $x \in \mathbb{R}_+$,

$$\varphi(b) = \begin{cases} y_1^\alpha & b = 0, \\ y_2^\alpha b^{s\alpha} + y_3^\alpha & b > 0. \end{cases}$$

If $y_2^\alpha \geq 0^\alpha$ and $0^\alpha \leq y_3^\alpha \leq y_1^\alpha$, then $\varphi \in GK_s^2$.

Proposition 5.1. Let $0 < y_1 < y_2$ and $s \in (0, 1)$. Then

$$\left| \frac{\Gamma(1+2\alpha)}{2^\alpha} A^s(y_1, y_2) - \frac{\Gamma(1+2\alpha) [\Gamma(1+\alpha)]^2}{2^\alpha (y_2 - y_1)^\alpha} L_s^s(y_1, y_2) \right| \\ \leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left| \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)} \right| \left\{ \frac{2^\alpha \Gamma(1+3\alpha) [\Gamma(1+\alpha)]^2}{\Gamma(1+4\alpha) \Gamma(1+2\alpha)} \right. \\ \left. + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{2^\alpha \Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \right\} \left[|y_1|^{(s-2)\alpha} + |y_2|^{(s-2)\alpha} \right]$$

Proof. The result follows from Remark 4.7 (2) with $\varphi : [0, 1] \longrightarrow [0^\alpha, 1^\alpha]$, $\varphi(x) = x^{s\alpha}$. and when $\alpha = 1$, we have the following inequality:

$$\left| A^s(y_1, y_2) - \frac{1}{y_2 - y_1} L_s^s(y_1, y_2) \right| \leq \frac{(y_2 - y_1)^2 |s(s-1)|}{48} \{ |y_1|^{s-2} + |y_2|^{s-2} \} \quad (5.1)$$

□

Proposition 5.2. *Let $0 < y_1 < y_2$ and $s \in (0, 1)$. Then*

$$\begin{aligned} & \left| \frac{\Gamma(1+2\alpha)}{2^\alpha} A^s(y_1, y_2) - \frac{\Gamma(1+2\alpha) [\Gamma(1+\alpha)]^2}{2^\alpha (y_2 - y_1)^\alpha} L_s^s(y_1, y_2) \right| \\ & \leq \frac{(y_2 - y_1)^{2\alpha}}{16^\alpha} \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right]^{\frac{1}{p_2}} \left| \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)} \right| \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)} \right]^{\frac{1}{p_1}} \\ & \times \left[\left(\left| \frac{y_1 + y_2}{2} \right|^{(s-2)p_2\alpha} + |y_1|^{(s-2)p_2\alpha} \right)^{\frac{1}{p_2}} + \left(\left| \frac{y_1 + y_2}{2} \right|^{(s-2)p_2\alpha} + |y_2|^{(s-2)p_2\alpha} \right)^{\frac{1}{p_2}} \right], \end{aligned}$$

where $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Proof. The result follows (4.9) with $\varphi : [0, 1] \longrightarrow [0^\alpha, 1^\alpha]$, $\varphi(x) = x^{s\alpha}$, and when $\alpha = 1$, we have the following inequality:

$$\begin{aligned} & \left| A^s(y_1, y_2) - \frac{1}{y_2 - y_1} L_s^s(y_1, y_2) \right| \\ & \leq \frac{(y_2 - y_1)^2 |s(s-1)|}{2^{\frac{1}{p_2}} 16 (2p_1 + 1)^{\frac{1}{p_1}}} \left\{ \left(\left| \frac{y_1 + y_2}{2} \right|^{(s-2)p_2} + |y_1|^{(s-2)p_2} \right)^{\frac{1}{p_2}} \right. \\ & \quad \left. + \left(\left| \frac{y_1 + y_2}{2} \right|^{(s-2)p_2} + |y_2|^{(s-2)p_2} \right)^{\frac{1}{p_2}} \right\}. \quad (5.2) \end{aligned}$$

□

Where $A(y_1, y_2)$ and $L_n(y_1, y_2)$ in (5.1) and (5.2) are known as

1. Arithmetic mean:

$$A(y_1, y_2) = \frac{y_1 + y_2}{2}, y_1, y_2 \in \mathbb{R}^+;$$

2. Logarithmic mean :

$$L(y_1, y_2) = \frac{y_1 - y_2}{\ln|y_1| - \ln|y_2|}, |y_1| \neq y_2, y_1, y_2 \neq 0, y_1, y_2 \in \mathbb{R}^+;$$

Generalized Log-mean:

$$L_n(y_1, y_2) = \left[\frac{y_2^{n+1} - y_1^{n+1}}{(n+1)(y_2 - y_1)} \right]^{\frac{1}{n}}, n \in \mathbb{Z} \setminus \{-1, 0\}, y_1, y_2 \in \mathbb{R}^+.$$

Now, we give application to wave equation on Cantor sets:

the wave equation on Cantor sets (local fractional wave equation) was given by [27]

$$\frac{\partial^{2\alpha} f(x, t)}{\partial t^{2\alpha}} = A^{2\alpha} \frac{\partial^{2\alpha} f(x, t)}{\partial x^{2\alpha}} \quad (5.3)$$

Following (5.3), a wave equation on Cantor sets was proposed as follows [29]:

$$\frac{\partial^{2\alpha} f(x, t)}{\partial t^{2\alpha}} = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha} f(x, t)}{\partial x^{2\alpha}}, \quad 0 \leq \alpha \leq 1 \quad (5.4)$$

where $f(x, t)$ is a fractal wave function and the initial value is given by $f(x, 0) = \frac{x^\alpha}{\Gamma(1+\alpha)}$. The solution of (5.4) is given as

$$f(x, t) = \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

By using (4.4), we have

$$\begin{aligned} & \frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{2^\alpha(y_2 - y_1)^\alpha} \int_{y_1}^{y_2} f(x, t)(dt)^\alpha - \frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(x, \frac{y_1 + y_2}{2}\right) \\ &= \frac{(y_2 - y_1)^\alpha}{8^\alpha \Gamma(1+2\alpha)} \left[\left(\frac{2}{y_2 - y_1} \right)^{2\alpha} {}_{y_1} I_{\frac{y_2 + y_1}{2}}^{(\alpha)} (t - y_1)^{2\alpha} x^{2\alpha} \frac{\partial^{2\alpha} f(x, t)}{\partial x^\alpha} \right. \\ & \quad \left. + {}_{y_1} I_{\frac{y_2 + y_1}{2}}^{(\alpha)} \left(\frac{2(t - y_1)}{y_2 - y_1} - 1 \right)^{2\alpha} x^{2\alpha} \frac{\partial^{2\alpha} f(x, t)}{\partial x^\alpha} \right] \end{aligned}$$

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